Student Workbook

Applying Double-Angle and Half-Angle Formulas

Name: ______________________ Date: ______________ Class: ________

ESSENTIAL QUESTION

How do the double-angle and half-angle formulas extend the sum/difference identities, and why are they so useful in calculus?

Lesson Overview

The double-angle formulas are special cases of the sum formulas with $\alpha=\beta=\theta$. The half-angle formulas are derived by solving the $\cos(2\theta)$ identities for $\sin\theta$ or $\cos\theta$. Power-reduction formulas rewrite $\sin^2\theta$ and $\cos^2\theta$ in terms of $\cos(2\theta)$ — essential for integrating powers of sine and cosine in calculus.

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

Cosine 2θ	$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$
Half-Angle	$\sin(\theta/2)=\pm\sqrt{(1-\cos\theta)/2}$; $\cos(\theta/2)=\pm\sqrt{(1+\cos\theta)/2}$; sign set by quadrant of $\theta/2$
Power Reduction	$\sin^2\theta = (1-\cos 2\theta)/2$; $\cos^2\theta = (1+\cos 2\theta)/2$

Worked Examples

EXAMPLE 1

Given $\sin\theta=3/5$ and θ is in QI, find $\sin(2\theta)$ and $\cos(2\theta)$

- Find $\cos\theta$: $\cos^2\theta = 1 - 9/25 = 16/25 \rightarrow \cos\theta = 4/5$ (QI)
- $\sin(2\theta) = 2\sin\theta\cos\theta = 2(3/5)(4/5) = 24/25$
- $\cos(2\theta) = \cos^2\theta - \sin^2\theta = 16/25 - 9/25 = 7/25$

Answer: $\sin(2\theta) = 24/25$, $\cos(2\theta) = 7/25$

EXAMPLE 2

Find the exact value of $\sin(\pi/8)$

- Use $\sin(\theta/2) = \pm\sqrt{(1-\cos\theta)/2}$ with $\theta = \pi/4$; $\cos(\pi/4) = \sqrt{2}/2$
- $\sin(\pi/8) = \sqrt{(1-\sqrt{2}/2)/2} = \sqrt{(2-\sqrt{2})/4} = \sqrt{2-\sqrt{2}}/2$
- Since $\pi/8$ is in QI, the sign is positive

Answer: $\sin(\pi/8) = \sqrt{2 - \sqrt{2}}/2$

EXAMPLE 3

Simplify: $2\sin(3x)\cos(3x)$

- Recognize the double-angle pattern: $2\sin\theta\cos\theta = \sin(2\theta)$ with $\theta = 3x$

Answer: $\sin(6x)$

EXAMPLE 4

Use a power-reduction formula to rewrite $\sin^2 x \cos^2 x$ as a function of $\cos(4x)$

- $\sin^2 x \cos^2 x = (\sin x \cos x)^2 = (\sin(2x)/2)^2 = \sin^2(2x)/4$
- Apply power-reduction to $\sin^2(2x)$: $\sin^2(2x) = (1 - \cos(4x))/2$

Answer: $\sin^2 x \cos^2 x = (1 - \cos(4x))/8$

EXAMPLE 5

Given $\cos\theta=-1/3$ and $\pi<\theta<3\pi/2$ (QIII), find $\cos(\theta/2)$

- $\cos(\theta/2) = \pm\sqrt{((1+\cos\theta)/2)} = \pm\sqrt{((1-1/3)/2)} = \pm\sqrt{((2/3)/2)} = \pm\sqrt{(1/3)} = \pm\sqrt{3}/3$
- θ is in QIII ($\pi<\theta<3\pi/2$), so $\pi/2<\theta/2<3\pi/4$ (QII)
- In QII, cosine is negative

Answer: $\cos(\theta/2) = -\sqrt{3}/3$

Guided Practice

Work through each problem below. Use the hint if you get stuck — write your final answer on the last line.

1 Given $\cos\theta=-4/5$ (QII), find $\sin(2\theta)$ and $\cos(2\theta)$.

Hint: Find $\sin\theta$ first (positive in QII), then apply the double-angle formulas.

2 Find the exact value of $\cos(\pi/8)$.

Hint: Use the half-angle formula for cosine with $\theta=\pi/4$. The angle $\pi/8$ is in QI, so the sign is positive.

3 Simplify: $\cos^2(5x) - \sin^2(5x)$

Hint: Recognize this as the double-angle form of cosine: $\cos^2\theta-\sin^2\theta=\cos(2\theta)$ with $\theta=5x$.

4 Rewrite $\cos^4 x$ using power-reduction formulas.

Hint: Write $\cos^4 x=(\cos^2 x)^2$, apply the power-reduction formula for $\cos^2 x$, then expand and apply it again.

5 Given $\sin\theta=-5/13$ (QIII), find $\tan(\theta/2)$.

Hint: Use $\tan(\theta/2)=\sin\theta/(1+\cos\theta)$. Find $\cos\theta$ first (negative in QIII).

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Key Vocabulary

Double-Angle Formula	A formula giving the trig value of 2θ in terms of the trig values of θ — a special case of the sum formulas with α=β=θ.
Half-Angle Formula	A formula giving the trig value of θ/2 in terms of the trig value of θ, derived from the cos(2θ) identities.
Power-Reduction Formula	A formula that rewrites $\sin^2\theta$ or $\cos^2\theta$ as a linear expression in cos(2θ), lowering the exponent.
Ambiguous Sign (±)	In half-angle formulas, the sign is resolved by identifying the quadrant of θ/2, not θ.

Check Your Understanding

Circle the letter of the correct answer for each question.

1 **sin(2θ) equals:** Multiple Choice

A $\sin^2\theta - \cos^2\theta$

B 2sinθcosθ

C sinθ + cosθ

D $2\sin^2\theta$

2 **Which form of cos(2θ) is most useful when you know sinθ?** Multiple Choice

A $\cos^2\theta - \sin^2\theta$

B $2\cos^2\theta - 1$

C $1 - 2\sin^2\theta$

D 2sinθcosθ

3 **The power-reduction formula for $\sin^2\theta$ is:** Multiple Choice

A (1 + cos2θ)/2

B (1 − cos2θ)/2

C (1 − cosθ)/2

D (1 + cosθ)/2

4 **If sinθ = 1/2 (QI), then sin(2θ) =** Multiple Choice

A 1/2

B √3/2

C √3/4

D 1/4

5 **The sign in the half-angle formula sin(θ/2)=±√((1−cosθ)/2) is determined by:** Multiple Choice

A The sign of sinθ

B The sign of cosθ

C The quadrant of θ/2

D The quadrant of θ

Common Mistakes to Avoid

✗ Writing $\sin(2\theta) = 2\sin\theta$ (forgetting the $\cos\theta$ factor).	✓ $\sin(2\theta) = 2\sin\theta\cos\theta$. Both factors are required.
✗ Using the wrong form of $\cos(2\theta)$ — e.g., using $\cos^2\theta-\sin^2\theta$ when only $\sin\theta$ is known.	✓ Choose the form that matches what you know: use $1-2\sin^2\theta$ when you know $\sin\theta$, and $2\cos^2\theta-1$ when you know $\cos\theta$.
✗ Determining the ± sign in a half-angle formula based on the quadrant of θ instead of θ/2.	✓ The sign is determined by the quadrant of θ/2. If θ=270° (QIII), then θ/2=135° (QII), where cosine is negative.
✗ Forgetting that the power-reduction formula uses 2θ: $\sin^2\theta = (1-\cos(2\theta))/2$.	✓ The argument inside cosine is 2θ, not θ. This is a very common error.

Math Tips

★ The three forms of $\cos(2\theta)$ are all equivalent — choose the one that eliminates the unknown. Know $\sin\theta$? Use $1-2\sin^2\theta$. Know $\cos\theta$? Use $2\cos^2\theta-1$.

★ Power-reduction formulas are the key to integrating $\sin^2x$ and $\cos^2x$ in calculus. Memorize them now and save hours later.

★ For half-angle problems, always determine the quadrant of θ/2 before choosing the ± sign — not the quadrant of θ.

★ $2\sin\theta\cos\theta = \sin(2\theta)$ works in reverse too: whenever you see a product of $\sin\theta$ and $\cos\theta$, collapse it into a single sine function.

★ $\tan(\theta/2) = \sin\theta/(1+\cos\theta) = (1-\cos\theta)/\sin\theta$ — both forms are equivalent. Use whichever avoids a zero denominator.