# Student Workbook

Using Sum and Difference Formulas



Name: ______________________ Date: ______________ Class: ________

**ESSENTIAL QUESTION**

How can you find the exact value of a trig function at an angle that is not on the standard unit circle, by expressing it as a sum or difference of familiar angles?

## Lesson Overview

The sum and difference formulas let you evaluate trig functions at angles formed by adding or subtracting two known angles, such as 15° or 75°. Strategy: express the target angle as a sum or difference of angles whose exact values you know — 30°, 45°, 60°, 90°, 120°, 135°, 150°, 180° (and their radian equivalents). Memory trick: sine keeps the same sign throughout; cosine uses the opposite sign.

$$\sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

| | |
|---|---|
| **Cosine** | $\cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$ (opposite sign) |
| **Tangent** | $\tan(\alpha\pm\beta) = (\tan\alpha \pm \tan\beta)/(1 \mp \tan\alpha\tan\beta)$ |
| **Strategy** | Write the target angle as a sum/difference of 30°, 45°, 60°, or 90°. |

## Worked Examples

**EXAMPLE 1**

**Find the exact value of sin(75°)**

- Write $75° = 45° + 30°$
- Apply $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$:
- $\sin(75°) = \sin45°\cos30° + \cos45°\sin30° = (\sqrt{2}/2)(\sqrt{3}/2) + (\sqrt{2}/2)(1/2) = \sqrt{6}/4 + \sqrt{2}/4$

**Answer: $\sin(75°) = (\sqrt{6} + \sqrt{2})/4$**

**EXAMPLE 2**

**Find the exact value of cos(15°)**

- Write $15° = 45° - 30°$
- Apply $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$:
- $\cos(15°) = \cos45°\cos30° + \sin45°\sin30° = (\sqrt{2}/2)(\sqrt{3}/2) + (\sqrt{2}/2)(1/2) = \sqrt{6}/4 + \sqrt{2}/4$

**Answer: $\cos(15°) = (\sqrt{6} + \sqrt{2})/4$**

**EXAMPLE 3**

**Find the exact value of $\tan(\pi/12)$**

- Note $\pi/12 = \pi/4 - \pi/6$
- Apply $\tan(\alpha-\beta) = (\tan\alpha-\tan\beta)/(1+\tan\alpha \tan\beta)$: $(1 - 1/\sqrt{3})/(1 + 1/\sqrt{3})$
- Multiply by $\sqrt{3}/\sqrt{3}$ and rationalize: $(\sqrt{3}-1)/(\sqrt{3}+1) = (4-2\sqrt{3})/2$

**Answer: $\tan(\pi/12) = 2 - \sqrt{3}$**

**EXAMPLE 4**

**Prove: $\cos(\pi/2 - \theta) = \sin\theta$**

- Apply $\cos(\alpha-\beta)$ with $\alpha = \pi/2$, $\beta = \theta$:
- $\cos(\pi/2-\theta) = \cos(\pi/2)\cos\theta + \sin(\pi/2)\sin\theta = 0\cdot\cos\theta + 1\cdot\sin\theta$

**Answer: Identity proved: $\cos(\pi/2 - \theta) = \sin\theta$**

**EXAMPLE 5**

**Given $\sin\alpha=3/5$ ($\alpha$ in QI) and $\cos\beta=-5/13$ ($\beta$ in QII), find $\sin(\alpha+\beta)$**

- Find $\cos\alpha$: $\cos^2\alpha = 1-9/25 = 16/25 \rightarrow \cos\alpha = 4/5$ (QI, positive)
- Find $\sin\beta$: $\sin^2\beta = 1-25/169 = 144/169 \rightarrow \sin\beta = 12/13$ (QII, positive)
- $\sin(\alpha+\beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta = (3/5)(-5/13) + (4/5)(12/13) = -15/65 + 48/65$

**Answer: $\sin(\alpha + \beta) = 33/65$**

## Guided Practice

Work through each problem below. Use the hint if you get stuck — write your final answer on the last line.

**1** Find the exact value of $\sin(105°)$.

*Hint: Write $105° = 60° + 45°$ and apply the sine sum formula.*

**2** Find the exact value of $\cos(\pi/12)$.

*Hint: Write $\pi/12 = \pi/4 - \pi/6$ and apply the cosine difference formula.*

**3** Prove: $\sin(\pi - \theta) = \sin\theta$

*Hint: Apply $\sin(\alpha-\beta)$ with $\alpha=\pi$, $\beta=\theta$. Use $\sin(\pi)=0$ and $\cos(\pi)=-1$.*

**4** Given $\cos\alpha=-4/5$ ($\alpha$ in QIII) and $\sin\beta=5/13$ ($\beta$ in QI), find $\cos(\alpha-\beta)$.

*Hint: Find $\sin\alpha$ (negative in QIII) and $\cos\beta$ (positive in QI), then apply the cosine difference formula.*

**5** Find the exact value of $\tan(165°)$.

*Hint: Write $165° = 120° + 45°$ and apply the tangent sum formula. $\tan(120°) = -\sqrt{3}$.*

## Key Vocabulary

| | |
|---|---|
| **Sum Formula** | A formula that gives the trig value of $\alpha+\beta$ in terms of the trig values of $\alpha$ and $\beta$ separately. |
| **Difference Formula** | A formula that gives the trig value of $\alpha-\beta$ in terms of the trig values of $\alpha$ and $\beta$ separately. |
| **Reference Angle Sum** | Expressing a non-standard angle as a sum/difference of 30°, 45°, 60°, or 90° (or radian equivalents). |
| **Cofunction Identity** | A special case of the difference formulas with $\alpha=\pi/2$, e.g. $\cos(\pi/2-\theta)=\sin\theta$. |
| **Rationalize** | Rewrite a fraction to remove a radical from the denominator, typically by multiplying by a conjugate. |
| **Quadrant Sign** | The + or − sign a trig value takes based on which quadrant the angle lies in (ASTC rule). |

## Check Your Understanding

Circle the letter of the correct answer for each question.

**1** **Which formula gives $\cos(\alpha + \beta)$?** Multiple Choice

A $\cos\alpha \cos\beta + \sin\alpha \sin\beta$

B $\cos\alpha \cos\beta - \sin\alpha \sin\beta$

C $\sin\alpha \cos\beta + \cos\alpha \sin\beta$

D $\sin\alpha \cos\beta - \cos\alpha \sin\beta$

**2** **$\sin(75°)$ expressed as a sum is:** Multiple Choice

A $\sin(30° + 45°)$

B $\sin(60° + 15°)$

C $\sin(90° - 15°)$

D All of the above

**3** **$\cos(\pi/2 - \theta)$ equals:** Multiple Choice

A $\cos\theta$

B $-\cos\theta$

C $\sin\theta$

D $-\sin\theta$

**4** **To find sin(α + β), you need:** Multiple Choice

- **A** Only sinα and sinβ
- **B** sinα, cosα, sinβ, and cosβ
- **C** Only cosα and cosβ
- **D** tanα and tanβ

**5** **The exact value of cos(15°) is:** Multiple Choice

- **A** $(\sqrt{6} - \sqrt{2})/4$
- **B** $(\sqrt{6} + \sqrt{2})/4$
- **C** $(\sqrt{3} + 1)/4$
- **D** $(\sqrt{3} - 1)/4$

## Common Mistakes to Avoid

| | |
|---|---|
| ✗ Writing cos(α + β) = cosα + cosβ (distributing cosine over addition). | ✓ cos(α+β) = cosα cosβ − sinα sinβ. Trig functions do NOT distribute over addition. |
| ✗ Using the wrong sign in the cosine formula: writing cos(α+β) = cosα cosβ + sinα sinβ. | ✓ Cosine uses the opposite sign: cos(α+β) uses −, and cos(α−β) uses +. |
| ✗ Forgetting to find all four values (sinα, cosα, sinβ, cosβ) before applying the formula. | ✓ Always determine all four values first, paying attention to the quadrant to get the correct sign. |
| ✗ Choosing the wrong quadrant sign for cosα or sinβ when given limited information. | ✓ Use the quadrant given. In QII, cosine is negative; in QIII, both sine and cosine are negative. |

## Math Tips

★ Memorize the six formulas as three pairs. Sine: same sign throughout. Cosine: opposite sign. Tangent: same sign on top, opposite on bottom.

★ The cofunction identities (sin(π/2−θ)=cosθ, etc.) are special cases of the difference formulas with α=π/2.

★ When finding exact values, always check whether the target angle can be written as a sum or difference of 30°, 45°, or 60° (or their multiples).

★ After computing, rationalize any denominator with a radical to put the answer in standard form.

★ For the tangent sum/difference formula, the denominator is 1 ∓ tanα tanβ — the sign is opposite to the angle operation.